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Zhu's algebra and the C_2-algebra in the symplectic and the orthogonal cases

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Abstract
We prove that Zhu's algebra and the C_2-algebra of type $\mathtt{C}_m$ have the same dimension, and we compute the graded character of the latter. Maximal parabolic subalgebras of the symplectic algebra play a central role in our construction. For the orthogonal algebras our methods do not allow us to describe the whole C_2-algebras; we get only a description of a certain quotient of the algebra.

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Introduction

In this paper, we continue the study of Zhu's algebras and C_2-algebras of WZW models, initiated in [FFL]. We briefly recall the setup.

The theory of vertex operator algebras (VOAs) plays a key role in the mathematical description of the structures arising in the conformal field theories (see [BF, GabGod, K2]). In particular spaces of states, partition functions and amplitudes can be described via the structure theory of vertex operator algebras and their representation theory.

The representation theory of a vertex operator algebra $\mathcal{V}$ is, in general, very complicated. But in some special cases (so-called rational VOAs) it is controlled by a certain finite-dimensional semisimple associative algebra $A(\mathcal{V})$, called Zhu's algebra (see [FZ, Z]). More precisely, irreducible representations of $A(\mathcal{V})$ are in one-to-one correspondence with irreducible representations of $\mathcal{V}$. Zhu's algebra can be explicitly computed in some special cases (for example for minimal models and WZW models). In this paper, we only deal with the WZW models associated with a simple Lie algebra $\mathfrak{g}$ on the non-negative integer level k. The corresponding Zhu's algebra (denoted by $A(\mathfrak{g}; k)$) is given by (see [FZ])

$$A(\mathfrak{g}; k) = U(\mathfrak{g}) \big/ \langle e_\theta^{k+1} \rangle,$$

where θ denotes the highest root, $e_\theta \in \mathfrak{g}$ is a corresponding root vector and $\langle e_\theta^{k+1}\rangle$ is the two-sided ideal in the universal enveloping algebra of $\mathfrak{g}$ generated by e_θ^{k+1}.

As mentioned above, the rationality is a very important characterization of a VOA. There exists a condition (called the C_2-cofiniteness condition, see [CF, M1, M2, Z]) which guaranties the rationality. Namely, the C_2-algebra $A_{[2]}(\mathcal{V})$ attached to $\mathcal{V}$ is defined as a quotient of $\mathcal{V}$ by the linear span of the elements of the form $a_n b$, $n \leqslant -2$, with $a, b \in \mathcal{V}$ and a_n being Fourier modes of the field corresponding to a. The C_2-cofiniteness condition reads as $\dim A_{[2]}(\mathcal{V}) < \infty$. The algebras $A(\mathcal{V})$ and $A_{[2]}(\mathcal{V})$ are very closely related (see [GG, GabGod, GN, N]). For WZW models, one has

$$A_{[2]}(\mathfrak{g}; k) = S^\bullet(\mathfrak{g})\big/\big\langle U(\mathfrak{g}) \circ e_\theta^{k+1}\big\rangle,$$

where $S^\bullet(\mathfrak{g})$ denotes the symmetric algebra of $\mathfrak{g}$, '$\circ$' denotes the action of $\mathfrak{g}$ on $S^\bullet(\mathfrak{g})$ induced by the adjoint action, $U(\mathfrak{g}) \circ e_\theta^{k+1} \subset S^\bullet(\mathfrak{g})$ is the (irreducible) $\mathfrak{g}$-module generated by $e_\theta^{k+1} \in S^{k+1}(\mathfrak{g})$, and $\langle U(\mathfrak{g}) \circ e_\theta^{k+1}\rangle$ is the ideal generated by this subspace.

It turned out that in several cases Zhu's algebra and the C_2-algebra have the same dimension (see [FFL, GG]). We note, however, that in general $A_{[2]}(\mathcal{V})$ can be viewed as a degeneration of $A(\mathcal{V})$ and thus may be bigger (this happens for example in the case of the WZW model of type E_8 with $k = 1$). We also note that the C_2-algebra has an extra grading missing in Zhu's algebra. For example, for $A_{[2]}(\mathfrak{g}; k)$ this grading is inherited from the degree grading on $S^\bullet(\mathfrak{g})$. It is therefore natural to ask for the graded dimension (or graded character) of the C_2-algebra.

In [GG], Gaberdiel and Gannon conjectured an explicit formula for the graded character of $A_{[2]}(\mathfrak{g}; k)$ for $\mathfrak{g} = \mathfrak{sl}_n$. They also conjectured that for all n and k one has $\dim A(\mathfrak{sl}_n; k) = \dim A_{[2]}(\mathfrak{sl}_n; k)$. These conjectures were proved in [FFL]. In general, we have the following conjecture.

Conjecture 0.1. *For all classical Lie algebras and all $k \geqslant 0$, one has*

$$\dim A(\mathfrak{g}; k) = \dim A_{[2]}(\mathfrak{g}; k).$$

In this paper we prove the conjecture for $\mathfrak{g}$ of type $\mathtt{C}$ ($\mathfrak{g} = \mathfrak{sp}_{2m}$) and compute the graded character of $A_{[2]}(\mathfrak{sp}_{2m}; k)$.

Our main idea is to identify the C_2-algebra $A_{[2]}(\mathfrak{sp}_{2m}; k)$ with the irreducible representation $\mathbb{V}(k\omega_{2m})$ of the bigger algebra $\mathfrak{sp}_{4m}$. We then use the geometry of the affine cone over the flag variety Sp_{4m}/P_{2m} and the restriction formulas by Littlewood, Koike and Terada [KT, Lw] to prove the graded character formula. Here $P_{2m} \subset Sp_{4m}$ is the parabolic subgroup associated with the fundamental weight ω_{2m}.

Unfortunately, our methods do not generalize to the orthogonal Lie algebras. The C_2-algebra $A_{[2]}(\mathfrak{so}_n; k)$ turns out to be bigger than the corresponding representation of $\mathfrak{so}_{2n}$ and, therefore, we can only describe a quotient of the C_2-algebra. We still show that this quotient (as a $\mathfrak{g}$-module) can be very naturally identified with a certain subspace of Zhu's algebra.

We close the introduction with the following remark. In [GabGod] (see also [GG, GN]), a new approach to confromal field theories, based on the spaces of correlation functions (amplitudes), is developed. These spaces can be defined in terms of quotients of spaces of states attached to a given conformal field theory (CFT). For instance, Zhu's algebras and the C_2-algebras are particular cases of these quotients. In general, the task to describe the spaces of correlation functions is very complicated. However, we hope that for WZW theories our approach, based on the algebra-geometric constructions from the representation theory of simple Lie algebras, can be useful.

The paper is organized as follows. In section 1, we recall the definitions of Zhu's algebra and the C_2-algebra and formulate the problem. In sections 2–4, we work out the case of the symplectic algebra $\mathfrak{sp}_{2m}$. In section 2, we describe the connection between the C_2-algebra $A_{[2]}(\mathfrak{sp}_{2m}; k)$ and certain representations of $\mathfrak{sp}_{4m}$. In section 3, we compute the graded character of $A_{[2]}(\mathfrak{sp}_{2m}; k)$. In section 4, we prove conjecture 0.1 in type C_m. Finally, in section 5, we discuss the case of the orthogonal algebras.

1. Zhu's algebra and the C_2-algebra

1.1. Definitions

Let $\mathfrak{g}$ be a simple Lie algebra. Let θ be the highest root of $\mathfrak{g}$ and $e_\theta \in \mathfrak{g}$ be a highest weight vector in the adjoint representation. Fix a non-negative integer k. Let $P_k^+(\mathfrak{g})$ be the set of level k integrable $\mathfrak{g}$ weights, i.e. the set of dominant integral $\mathfrak{g}$ weights λ satisfying $(\lambda, \theta) \leqslant k$ (the Killing form has been normalized by the usual condition $(\theta, \theta) = 2$). The following theorem 1.1 is proved in [FZ].

Theorem 1.1. *The level k Zhu algebra $A(\mathfrak{g}; k)$ is the quotient of the universal enveloping algebra $U(\mathfrak{g})$ by the two-sided ideal generated by e_θ^{k+1}:*

$$A(\mathfrak{g}; k) = U(\mathfrak{g}) \big/ \big\langle e_\theta^{k+1} \big\rangle.$$

In addition, one has an isomorphism of $\mathfrak{g}$-modules:

$$A(\mathfrak{g}; k) \simeq \bigoplus_{\lambda \in P_k^+(\mathfrak{g})} V(\lambda) \otimes V(\lambda)^*.$$

The form of the description of $A(\mathfrak{g}; k)$ arises ultimately because of the Peter–Weyl theorem.

Notation 1.2. *Let $S(\mathfrak{g}) = \bigoplus_{m=0}^{\infty} S^m(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$. For $v \in S^m(\mathfrak{g})$ and $a \in \mathfrak{g}$, let $av \in S^{m+1}(\mathfrak{g})$ be the product in the symmetric algebra. Each homogeneous summand $S^m(\mathfrak{g})$ is a $\mathfrak{g}$-module by the adjoint action on $\mathfrak{g}$. For $v \in S^m(\mathfrak{g})$ and $a \in \mathfrak{g}$, we denote by $a \circ v \in S^m(\mathfrak{g})$ the adjoint action of a.*

The C_2-algebra associated with $\mathcal{V}(\mathfrak{g}; k)$ can be described as follows. The level k C_2-algebra $A_{[2]}(\mathfrak{g}; k)$ is the quotient of the symmetric algebra $S(\mathfrak{g})$ by the ideal generated by the subspace $V_{k+1} = U(\mathfrak{g}) \circ e_\theta^{k+1} \hookrightarrow S^{k+1}(\mathfrak{g})$:

$$A_{[2]}(\mathfrak{g}; k) = S(\mathfrak{g}) / \langle V_{k+1} \rangle.$$

Remark 1.3. The subspace $V_{k+1} \hookrightarrow S^{k+1}(\mathfrak{g})$ is isomorphic to the irreducible $\mathfrak{g}$-module $V((k+1)\theta)$ of highest weight $(k+1)\theta$. The algebra $A_{[2]}(\mathfrak{g}; k)$ is naturally a $\mathfrak{g}$-module, the module structure being induced by the adjoint action. Note that $A_{[2]}(\mathfrak{g}; k)$ is not a $\mathfrak{g} \oplus \mathfrak{g}$-module, differently from $A(\mathfrak{g}; k)$.

Consider the standard filtration $F_\bullet$ on the universal enveloping algebra $U(\mathfrak{g})$, such that $\mathrm{gr}_\bullet F \simeq S(\mathfrak{g})$. Let $F_\bullet(k)$ be the induced filtration on the quotient algebra $A(\mathfrak{g}; k)$. We have an obvious surjection

$$A_{[2]}(\mathfrak{g}; k) \to \mathrm{gr}_\bullet F(k). \tag{1.1}$$

Therefore, we have a surjective homomorphism of $\mathfrak{g}$-modules

$$A_{[2]}(\mathfrak{g}; k) \to A(\mathfrak{g}; k) \tag{1.2}$$

and thus $\dim A_{[2]}(\mathfrak{g}; k) \geqslant \sum_{\beta \in P_k^+(\mathfrak{g})} (\dim V_\beta)^2$. A natural question is: when does this inequality turn into an equality? In this paper, we are also interested in the degree grading on $A_{[2]}(\mathfrak{g}; k)$ and in the corresponding graded decomposition into the direct sum of $\mathfrak{g}$-modules. Let

$$S(\mathfrak{g}) = \bigoplus_{m \geqslant 0} S^m(\mathfrak{g})$$

be the degree of decomposition of the symmetric algebra. This decomposition induces the decomposition of the C_2-algebra:

$$A_{[2]}(\mathfrak{g}; k) = \bigoplus_{m \geqslant 0} A^m_{[2]}(\mathfrak{g}; k). \tag{1.3}$$

Each space $A^m_{[2]}(\mathfrak{g}; k)$ is naturally a representation of $\mathfrak{g}$. Our main questions are as follows.

(i) Prove the equality of the dimensions of $A_{[2]}(\mathfrak{g}; k)$ and $A(\mathfrak{g}; k)$.
(ii) Find the decomposition of $A^m_{[2]}(\mathfrak{g}; k)$ into the direct sum of irreducible $\mathfrak{g}$-modules.

Recall that the case $\mathfrak{g} = \mathfrak{sl}_n$ was considered in [GG] and [FFL]. To be more precise, it was conjectured in [GG] and proved in [FFL] that

$$\dim A_{[2]}(\mathfrak{sl}_n; k) = \dim A(\mathfrak{sl}_n; k)$$

and, as $\mathfrak{sl}_n$-modules (not as $\mathfrak{sl}_n \oplus \mathfrak{sl}_n$-modules, despite the description as tensor products), one has a decomposition

$$A^m_{[2]}(k) = \frac{\bigoplus_{\substack{\lambda:\ k \geqslant \lambda_1, \lambda_n \geqslant 0 \\ \lambda_1 + \cdots + \lambda_n = m}} V(\lambda) \otimes V(\lambda)^*}{\bigoplus_{\substack{\lambda:\ k-1 \geqslant \lambda_1, \lambda_n \geqslant 0 \\ \lambda_1 + \cdots + \lambda_n = m-1}} V(\lambda) \otimes V(\lambda)^*},$$

where the $\mathfrak{gl}_n$-module $V(\lambda)$ is regarded as $\mathfrak{sl}_n$-module with highest weight $(\lambda_1 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n)$.

In this paper, we solve problems (i) and (ii) for $\mathfrak{g} = \mathfrak{sp}_{2m}$ and obtain certain results for orthogonal algebras.

1.2. The general CFT-based approach

In this subsection, we recall how Zhu's algebra and the C_2-algebra show up in the context of conformal field theory on the sphere $\mathbb{P}^1$. Our main references are [DFMS] for general CFT-related questions and [BF, K2] for the theory of vertex operator algebras.

Let $\mathcal{H}$ be the vacuum space of states (Fock space) of a given CFT $\mathcal{V}$. The space $\mathcal{H}$ carries a degree grading with respect to the zeroth Virasoro operator L_0:

$$\mathcal{H} = \bigoplus_{k \geqslant 0} \mathcal{H}(k), \qquad L_0 a = ka \quad \forall a \in \mathcal{H}(k).$$

For instance, $\mathcal{H}(0)$ is spanned by a vacuum vector $|0\rangle$. In what follows for $a \in \mathcal{H}(k)$ we write $|a| = k$. The field-states correspondence attaches a field (vertex operator) $Y(a, z)$ to each vector $a \in \mathcal{H}$. For a vector $a \in \mathcal{H}(k)$, the modes decomposition of $Y(a, z)$ is given by $Y(a, z) = \sum_{m \in \mathbb{Z}} a_m z^{-m-k}$, where a_m are degree $-m$ linear operators on $\mathcal{H}$. Vertex operators $Y(a, z)$ satisfy the OPE, which describes the products $Y(a, z)Y(b, w)$.

Correlation functions (amplitudes) play a very important role in the CFT. An example of these functions is vacuum-to-vacuum correlators

$$\langle 0|Y(a_1, z_1) \cdots Y(a_l, z_l)|0\rangle.$$

In [GabGod], the authors suggested an axiomatic formulation of conformal field theories in terms of systems of amplitudes. Let V be a space of quasi-primary states that generate the theory. To each l-tuple $a_1, \ldots, a_l \in V$, one can attach a correlation function $\langle Y(a_1, z_1)\cdots Y(a_l, z_l)\rangle$. These amplitudes form a system with certain properties and describe the structure of the theory (irreducible highest weight representations, fusion rules, OPE, etc). Therefore, an important task is to construct examples (or even spaces of examples) of systems of correlation functions. These spaces are constructed as follows. To each collection of points $\mathbf{u} = (u_1, \ldots, u_k)$, one can attach a space $\mathcal{O}_{\mathbf{u}} \hookrightarrow \mathcal{H}$ such that the dual space $A^*_{\mathbf{u}} = (\mathcal{H}/\mathcal{O}_{\mathbf{u}})^*$ parameterizes the systems of correlation functions satisfying the so-called highest weight condition (depending on $\mathbf{u}$). The cases $\mathbf{u} = (\infty, -1)$ and $\mathbf{u} = (\infty, \infty)$ are of the special interest. The corresponding spaces $\mathcal{O}_{\mathbf{u}}$ are given by

$$\mathcal{O}_{(\infty,-1)} = \mathrm{span}\{\mathrm{Res}_z(Y(a,z)z^{-2}(1+z)^{|a|}b), a, b \in \mathcal{H}\},$$
$$\mathcal{O}_{(\infty,\infty)} = \mathrm{span}\{a_{-2}b, a, b \in \mathcal{H}\}.$$

The spaces $A_{(\infty,-1)}$ and $A_{(\infty,\infty)}$ can be endowed with multiplication and are referred to as Zhu's algebra and the C_2-algebra (see [Z]). We give some remarks on their importance below.

One of the most important concepts in the CFT is rationality. A rational theory $\mathcal{V}$ has several finiteness properties: the number of irreducible highest weight representations is finite, the fusion coefficients are finite, the characters are convergent (for $|q| < 1$) and are closed under modular transformations. It turns out that the algebras $A(\mathcal{V}) = A_{(\infty,-1)}$ and $A_{[2]}(\mathcal{V}) = A_{(\infty,\infty)}$ carry crucial information about the rationality and about irreducible representations of $\mathcal{V}$. Conjecturally, the rationality of $\mathcal{V}$ is characterized by finite-dimensionality and semi-simplicity of $A(\mathcal{V})$. If both properties hold, then irreducible representations of $A(\mathcal{V})$ are in one-to-one correspondence with irreducible highest weight representations of $\mathcal{V}$. The algebra $A_{[2]}(\mathcal{V})$ is a degeneration of Zhu's algebra. Therefore it can be *a priori* 'bigger' than $A(\mathcal{V})$. It was shown in [Z] that the finite dimensionality of $\mathcal{V}$ (the so-called C_2-condition) implies that the number of irreducible representations of $\mathcal{V}$ is finite and that the fusion coefficients are finite (note, however, that it does not imply the rationality of $\mathcal{V}$, see [GK]).

In this paper, we are only concerned with the WZW theories associated with simple Lie algebra $\mathfrak{g}$ and non-negative integer level k. In this case, the vacuum space of states $\mathcal{H}$ is isomorphic to the level k vacuum (with trivial highest weight with respect to $\mathfrak{g} \otimes 1$) representation of the affine Kac–Moody Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ (see [K1]). The precise description of Zhu's algebra in this case was given in [FZ] (see subsection 1.1 above). We note that the C_2-algebra is given by

$$A_{[2]}(\mathcal{V}) = \mathcal{H}/\mathrm{span}\{(x \otimes t^{-n})v :\ x \in \mathfrak{g}, v \in \mathcal{H}, n \geqslant 2\}.$$

2. The Lie algebra $\mathfrak{sp}_{4m}$ and the C_2-algebra $A(\mathfrak{sp}_{2m}; k)$

The idea of the following construction is to realize $A_{[2]}(\mathfrak{sp}_{2m}; k)$ as a representation of the much larger algebra $\mathfrak{sp}_{4m}$, and then use restriction algorithm arguments to prove a graded character formula as well as the equality of the dimensions of the C_2-algebra and Zhu's algebra.

2.1. Sympletic algebras: generalities

The enumeration of the fundamental weights is as in [B]. Let $\omega_1, \ldots, \omega_m$ be the set of fundamental weights for the Lie algebra $\mathfrak{sp}_{2m}$. The highest root is $\theta = 2\omega_1$, and for a dominant integral $\lambda = \sum_{i=1}^m a_i\omega_i$ the condition $(\lambda, \theta) \leqslant k$ reads as $\sum_{i=1}^m a_i \leqslant k$. Recall that for any dominant weight $\lambda \in P^+(\mathfrak{sp}_{2m})$, we have $V(\lambda) \simeq V(\lambda)^*$, so theorem 1.1 can be reformulated as

$$A(\mathfrak{sp}_{2m}; k) \simeq \bigoplus_{\substack{\lambda=\sum_{i=1}^m a_i\omega_i \\ \sum_{i=1}^m a_i \leqslant k}} V(\lambda) \otimes V(\lambda). \tag{2.1}$$

Let $\mathfrak{p}_{2m} \subset \mathfrak{sp}_{4m}$ be the maximal parabolic Lie subalgebra associated with the fundamental weight ω_{2m}. We fix a Levi decomposition $\mathfrak{p}_{2m} = \mathfrak{l} \oplus \mathfrak{n}$ and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{p}_{2m}$, where $\mathfrak{l} \simeq \mathfrak{gl}_{2m}$ is the Lie algebra, $\mathfrak{n} \simeq S^2(\mathbb{C}^{2m})$ and $\mathfrak{n}^- \simeq S^2(\mathbb{C}^{2m})^*$ as the $\mathfrak{l} = \mathfrak{gl}_{2m}$-module. The $\mathfrak{gl}_{2m}$-representations on $\mathfrak{n}$ and $\mathfrak{n}^-$ remain irreducible with respect to the action of the subalgebra $\mathfrak{sp}_{2m} \subset \mathfrak{gl}_{2m}$, both representations are isomorphic to the adjoint representation of the symplectic Lie algebra. Summarizing we have

Lemma 2.1. *As $\mathfrak{l} = \mathfrak{gl}_{2m}$-module we have isomorphisms $\mathfrak{n} \simeq S^2(\mathbb{C}^{2m})$ and $\mathfrak{n}^- \simeq S^2(\mathbb{C}^{2m})^*$, and as $\mathfrak{sp}_{2m}$-module we have isomorphisms $\mathfrak{n} \simeq \mathfrak{n}^- \simeq \mathfrak{sp}_{2m}$.*

In the following, we always assume that for $\ell \in \mathbb{N}$ the symplectic group $Sp_{2\ell}$ is realized as the group leaving invariant the skew symmetric form on $\mathbb{C}^{2\ell}$ defined by the $2\ell \times 2\ell$-matrix:

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \iddots & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For an $m \times m$ matrix A, let A^{nt} be the transpose of a matrix with respect to the diagonal given by $i + j = 2m+1$, i.e. for $A = (a_{i,j})$ the matrix $A^{nt} = \left(a^{nt}_{i,j}\right)$ is given by $a^{nt}_{i,j} = a_{2m+1-j,2m+1-i}$. The Lie algebra of the symplectic group Sp_{2m} can then be described as the following set of matrices:

$$\mathfrak{sp}_{2m} = \left\{ \begin{pmatrix} A & B \\ C & -A^{nt} \end{pmatrix} \middle| A, B, C \in M_m, B = B^{nt}, C = C^{nt} \right\},$$

with the maximal torus $\mathfrak{t} = \mathrm{diag}(t_1, \ldots, t_m, -t_m \ldots, -t_1)$ and the Borel subalgebra being the upper triangular matrices of the form above.

The Lie algebra of the symplectic group $Sp_{2m} \subset GL_{2m}$ embedded in the Levi subgroup $GL_{2m} \subset Sp_{4m}$ can be seen as the set of matrices of the following form:

$$\mathfrak{sp}_{2m} = \left\{ \begin{pmatrix} A & B & 0 & 0 \\ C & -A^{nt} & 0 & 0 \\ 0 & 0 & A & -B \\ 0 & 0 & -C & -A^{nt} \end{pmatrix} \middle| \begin{matrix} A, B, C \in M_m, \\ B = B^{nt} \\ C = C^{nt} \end{matrix} \right\} \subset \mathfrak{sp}_{4m}. \tag{2.2}$$

There is also a maximal reductive sub-Lie algebra of type $\mathtt{C}_m + \mathtt{C}_m$ sitting inside $\mathfrak{sp}_{4m}$ in the following way:

$$\mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2m} = \left\{ \begin{pmatrix} K & 0 & 0 & L \\ 0 & X & Y & 0 \\ 0 & Z & -X^{nt} & 0 \\ M & 0 & 0 & -K^{nt} \end{pmatrix} \middle| \begin{matrix} K, L, M, X, Y, Z \in M_m \\ L = L^{nt}, M = M^{nt} \\ Y = Y^{nt}, Z = Z^{nt} \end{matrix} \right\}.$$

Let $\mathbb{I}_m$ denote the $m \times m$ identity matrix and let J' be the $2m \times 2m$ matrix of the form

$$J' = \begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0 \end{pmatrix}.$$

Inside the Lie subalgebra $\mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2m}$, we have a diagonally embedded symplectic Lie algebra $\Delta(\mathfrak{sp}_{2m})$, where.

$$\Delta : \mathfrak{sp}_{2m} \hookrightarrow \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2m}, \qquad Q \mapsto (Q, J' Q J'^{-1}), \tag{2.3}$$

which has the same maximal torus as the embedded symplectic Lie subalgebra $\mathfrak{sp}_{2m}$ described in (2.2). As a consequence, we see

Lemma 2.2. *For a $\mathfrak{sp}_{4m}$-representation $\mathbb{W}(\lambda)$ let*

$$\mathrm{res}^{\mathfrak{sp}_{4m}}_{\mathfrak{sp}_{2m}} \mathbb{W}(\lambda)$$

be the $\mathfrak{sp}_{2m}$-module obtained via the embedding in (2.2) and

$$\mathrm{res}^{\mathfrak{sp}_{4m}}_{\Delta(\mathfrak{sp}_{2m})} \mathbb{W}(\lambda)$$

be the $\mathfrak{sp}_{2m}$-module obtained via the embedding in (2.3). Then

$$\mathrm{res}^{\mathfrak{sp}_{4m}}_{\mathfrak{sp}_{2m}} \mathbb{W}(\lambda) \simeq \mathrm{res}^{\mathfrak{sp}_{4m}}_{\Delta(\mathfrak{sp}_{2m})} \mathbb{W}(\lambda).$$

2.2. The Lie algebra $\mathfrak{sp}_{4m}$ and $A_{[2]}(\mathfrak{sp}_{2m}; k)$

We fix the standard maximal torus (diagonal matrices) and Borel subalgebra (upper triangular matrices) for $\mathfrak{gl}_{2m}$. Then the maximal torus and Borel subalgebra of $\mathfrak{sp}_{2m}$ and $\mathfrak{gl}_{2m}$ are contained in each other. Let θ be the maximal root in the root system of $\mathfrak{sp}_{2m}$. We identify the module $\mathfrak{sp}_{2m}$ again with $\mathfrak{n}^-$ (lemma 2.1). Fix a highest root vector x_θ, then x_θ is a highest weight vector for the adjoint action of $\mathfrak{sp}_{2m}$ as well as for the irreducible action of $\mathfrak{gl}_{2m}$ on the same representation space. It follows that

$$x_\theta^{k+1} \in S^\bullet(\mathfrak{sp}_{2m}) = S^\bullet(S^2(\mathbb{C}^{2m}))^* = S^\bullet(\mathfrak{n}^-) \tag{2.4}$$

is a highest weight vector of weight $2(k+1)\omega_1$ for the action of the symplectic group Sp_{2m} and of weight $-2(k+1)\epsilon_{2m}$ for the general linear group GL_{2m}. One can check easily the following connection between x_θ and the root vectors for the Lie algebra $\mathfrak{sp}_{4m}$.

Lemma 2.3. *Let $X_{-\alpha_{2m}} \in \mathfrak{n}^- \subset \mathfrak{sp}_{4m}$ be a root vector for the negative of the simple root α_{2m} in the root system for the Lie algebra $\mathfrak{sp}_{4m}$. With respect to the embedding in (2.2), $X_{-\alpha_{2m}}$ is a weight vector for the Lie algebra $\mathfrak{sp}_{2m}$ of weight θ and hence can be identified with x_θ.*

To distinguish between the highest weight representations of the different groups, we write $V(\lambda)$ for the Sp_{2m}-representations, $\mathbb{V}(\lambda)$ for the GL_{2m}-representations and $\mathbb{W}(\lambda)$ for the Sp_{4m}-representations of highest weight λ (whenever this makes sense).

The irreducible GL_{2m}-module $U(\mathfrak{gl}_{2m}) \circ x_\theta^{k+1}$ generated by x_θ^{k+1} is the module $\mathbb{V}(2(k+1)\omega_1)^*$, and hence remains irreducible when restricted to Sp_{2m}, i.e. we have the following sequence of equalities of vector spaces:

$$\begin{aligned} U(\mathfrak{gl}_{2m}) \circ x_\theta^{k+1} &= U(\mathfrak{sp}_{2m}) \circ x_\theta^{k+1} \\ &= U(\mathfrak{sp}_{2m}) \circ X_{-\alpha_{2m}}^{k+1} \\ &= U(\mathfrak{gl}_{2m}) \circ X_{-\alpha_{2m}}^{k+1}. \end{aligned}$$

Let $\mathbb{W}(k\omega_{2m})$ be the irreducible Sp_{4m}-module of highest weight $k\omega_{2m}$. The nilpotent radical $\mathfrak{n}$ of $\mathfrak{p}_{2m}$ is Abelian (since ω_{2m} is a cominuscule weight, see for example [FFL]). Recall the following isomorphism of $\mathfrak{l}$-modules (see [FFL], lemma 3.1):

$$\mathbb{W}(k\omega_{2m}) \otimes \mathbb{C}_{-k\omega_{2m}} \simeq S^\bullet(\mathfrak{n}^-) \big/ \langle U(\mathfrak{l}) \circ x_\theta^{k+1} \rangle,$$

where $\langle\cdots\rangle$ denotes the ideal generated by the corresponding subspace. Combining this isomorphism with (2.4), we get as a consequence the following isomorphisms of GL_{2m}-modules as well as Sp_{2m}-modules. In particular, the C_2-algebra $A_{[2]}(\mathfrak{sp}_{2m};k)$ inherits the structure of a GL_{2m}-module.

Lemma 2.4.

$$\begin{aligned}\mathbb{W}(k\omega_{2m})\otimes\mathbb{C}_{-k\omega_{2m}} &\simeq S^\bullet(\mathfrak{n}^-)\big/\big\langle U(\mathfrak{gl}_{2m})\circ X^{k+1}_{-\alpha_{2m}}\big\rangle\\ &= S^\bullet(\mathfrak{sp}_{2m})\big/\big\langle U(\mathfrak{sp}_{2m})\circ x^{k+1}_{\theta}\big\rangle\\ &\simeq A_{[2]}(\mathfrak{sp}_{2m};k).\end{aligned}$$

3. The graded character of the C_2-algebra for $\mathfrak{sp}_{2m}$

Let $P_{2m}\subset Sp_{4m}$ be the parabolic subgroup associated with the fundamental weight ω_{2m}. By [L1] it is known that the action of the Levi subgroup $L=GL_{2m}\subset P_{2m}$ on Sp_{4m}/P_{2m} is spherical, i.e. a Borel subgroup of L has a dense orbit in Sp_{4m}/P_{2m}. As a consequence, the restriction of an irreducible Sp_{4m}-module of highest weight $\ell\omega_{2m}$, $\ell\in\mathbb{N}$, to L is multiplicity free. Let ω_0 denote the trivial character. The tables in [L1] imply

Proposition 3.1. *As the GL_{2m}-module, the C_2-algebra $A_{[2]}(\mathfrak{sp}_{2m};k)$ decomposes as*

$$A_{[2]}(\mathfrak{sp}_{2m};k)=\bigoplus_{\substack{\lambda=2(a_1\omega_1+\ldots+a_{2m-1}\omega_{2m-1})\\ a_1+\ldots+a_{2m}\leqslant k}}\mathbb{V}(\lambda)\otimes\mathbb{C}_{2(a_{2m}-k)\omega_{2m}}.$$

Proof. The lemma above and the decomposition formula in [L1] imply

$$\begin{aligned}\big(\mathrm{res}^{Sp_{4m}}_{GL_{2m}}\mathbb{W}(k\omega_{2m})\big)\otimes\mathbb{C}_{-k\omega_{2m}} &= \bigoplus_{\substack{\lambda=a_0\omega_0+a_12\omega_1+\ldots+a_{2m}2\omega_{2m}\\ a_0+a_1+\ldots+a_{2m}=k}}\mathbb{V}(\lambda)\otimes\mathbb{C}_{-2k\omega_{2m}}\\ &= \bigoplus_{\substack{\lambda=2(a_1\omega_1+\ldots+a_{2m-1}\omega_{2m-1})\\ a_1+\ldots+a_{2m}\leqslant k}}\mathbb{V}(\lambda)\otimes\mathbb{C}_{2(a_{2m}-k)\omega_{2m}}.\end{aligned}$$

□

The center Z of GL_{2m}, $Z:=\{t.\mathbb{I}_{2m}\mid t\in\mathbb{C}^*\}$, acts on $\mathfrak{sp}_{2m}=S^2(\mathbb{C}^{2m})^*=\mathfrak{n}^-$ by t^{-2} and hence on $S^j(\mathfrak{sp}_{2m})$ by t^{-2j}. For a Young diagram (or a partition) λ the action of the center on a representation $\mathbb{V}(\lambda)$ can be described as follows. The element $t.\,\mathbb{I}_{2m}\in Z$ acts as a multiplication by t to the power given by the number of boxes in the diagram. This leads to the following corollary.

Corollary 3.2. *The jth graded component of the C_2-algebra $A_{[2]}(\mathfrak{sp}_{2m};k)=\bigoplus_{j\geqslant 0}A^j_{[2]}(\mathfrak{sp}_{2m};k)$ decomposes as $\mathfrak{gl}_{2m}$-module as follows:*

$$A^j_{[2]}(\mathfrak{sp}_{2m};k)=\sum_{\substack{\lambda=2(a_1\omega_1+\ldots+a_{2m-1}\omega_{2m-1})\\ a_1+\ldots+a_{2m}\leqslant k\\ a_1+2a_2+3a_3+\ldots+(2m-1)a_{2m-1}+2ma_{2m}+j=2mk}}\mathbb{V}(\lambda)\otimes\mathbb{C}_{2(a_{2m}-k)\omega_{2m}}.$$

Proof. We know that $t.\mathbb{I}_{2m}\in Z$ acts on $\mathbb{V}(\lambda)\otimes\mathbb{C}_{2(a_{2m}-k)\omega_{2m}}$ as

$$t^{2(a_1+2a_2+\cdots+(2m-1)a_{2m-1})}t^{2m(2a_{2m}-2k)}.$$

Therefore, the condition

$$\mathbb{V}(\lambda)\otimes\mathbb{C}_{2(a_{2m}-k)\omega_{2m}}\hookrightarrow A^j_{[2]}(\mathfrak{sp}_{2m};k)$$

can be reformulated as

$$a_1 + 2a_2 + 3a_3 + \cdots + (2m-1)a_{2m-1} + 2ma_{2m} + j = 2mk,$$

which proves the corollary. □

The corollary provides a first step for a formula for the decomposition of $A^j_{[2]}(\mathfrak{sp}_{2m}; k)$ as $\mathfrak{sp}_{2m}$-module. To deduce from the above a graded character formula, recall the following restriction rule due to Littlewood [Lw], respectively, its generalization by Koike and Terada [KT]. In the GL_{2m} and Sp_{2m} settings, it is often convenient to use at the same time the language of partitions as well as the language of highest weights. By the abuse of notation we denote for a dominant weight $\lambda = \sum_{i=1}^{2m} a_i\omega_i$ by $\lambda = (\lambda_1, \lambda_2, \ldots)$ also the associated partition given by the rule

$$\lambda_1 = \sum_{i=1}^{2m} a_i, \qquad \lambda_2 = \sum_{i=2}^{2m} a_i, \qquad \ldots, \qquad \lambda_{2m-1} = a_{2m-1} + a_{2m}, \qquad \lambda_{2m} = a_{2m}.$$

For three such partitions λ, μ, ν we denote by $N^\lambda_{\mu,\nu}$ the associated Littlewood–Richardson coefficient, i.e. for a tensor product of GL_{2m}-representations, $N^\lambda_{\mu,\nu}$ is the multiplicity of $\mathbb{V}(\lambda)$ in the tensor product $\mathbb{V}(\mu) \otimes \mathbb{V}(\nu)$.

Note that for a partition λ with more than m parts a representation $V(\lambda)$ for the symplectic group Sp_{2m} is not defined. In [KT], section 2.4, one finds an algorithm (folding of Young diagrams) which associates with a partition λ a partition $\pi(\lambda)$ with less than or equal to m parts, and a sign which we denote by $\mathrm{sign}(\lambda)$. For such a partition, Koike and Terada define as the character associated with this partition $\mathrm{char} V(\lambda) := \mathrm{sign}(\lambda)\mathrm{char} V(\pi(\lambda))$. If λ has less than or equal to m parts, then the sign is plus and $\pi(\lambda) = \lambda$.

Theorem 3.3 ([KT]). *Suppose that λ is a partition having at most $2m$ positive parts. Then*

$$\mathrm{char}\big(\mathrm{res}^{GL_{2m}}_{Sp_{2m}} \mathbb{V}(\lambda)\big) = \sum_{\mu} \sum_{(2\nu)^t} N^\lambda_{\mu,(2\nu)^t} \mathrm{char} V(\mu),$$

where the sum is over all non-negative integer partitions μ and ν, and $(2\nu)^t$ denotes the transpose of the partition 2ν.

As a consequence we get the following.

Theorem 3.4. *The jth graded component of the C_2-algebra $A_{[2]}(\mathfrak{sp}_{2m}; k) = \bigoplus_{j\geqslant 0} A^j_{[2]}(\mathfrak{sp}_{2m}; k)$ decomposes as the Sp_{2m}-module as follows:*

$$\begin{aligned}\mathrm{char} A^j_{[2]}(\mathfrak{sp}_{2m}; k) &= \sum_{\substack{\lambda = 2(a_1\omega_1 + \ldots + a_{2m-1}\omega_{2m-1}) \\ a_1 + \ldots + a_{2m} \leqslant k \\ a_1 + 2a_2 + 3a_3 + \ldots + 2ma_{2m} + j = 2mk}} \mathrm{char}\big(\mathrm{res}^{GL_{2m}}_{Sp_{2m}} \mathbb{V}(\lambda)\big) \\ &= \sum_{\substack{\lambda = 2(a_1\omega_1 + \ldots + a_{2m-1}\omega_{2m-1}) \\ a_1 + \ldots + a_{2m} \leqslant k \\ a_1 + 2a_2 + 3a_3 + \ldots + 2ma_{2m} + j = 2mk}} \sum_{\mu} \sum_{(2\nu)^t} N^\lambda_{\mu,(2\nu)^t} \mathrm{char} V(\mu).\end{aligned}$$

4. Comparing dimensions in type C_m

Theorem 4.1. *The C_2-algebra $A_{[2]}(\mathfrak{sp}_{2m}; k)$ and Zhu's algebra $A(\mathfrak{sp}_{2m}; k)$ have the same dimension.*

Proof. The proof is divided into several steps.

Step 1. We transform the problem into a restriction problem for another group. By lemma 2.4, we have an isomorphism of Sp_{2m}-modules:

$$A_{[2]}(\mathfrak{sp}_{2m}; k) \simeq \mathrm{res}^{Sp_{4m}}_{Sp_{2m}} \mathbb{V}(k\omega_{2m}),$$

where the embedding $Sp_{2m} \hookrightarrow Sp_{4m}$ is given by (2.2). By lemma 2.2, we know that

$$\mathrm{res}^{\mathfrak{sp}_{4m}}_{\mathfrak{sp}_{2m}} \mathbb{V}(k\omega_{2m}) \simeq \mathrm{res}^{\mathfrak{sp}_{4m}}_{\Delta(\mathfrak{sp}_{2m})} \mathbb{V}(k\omega_{2m}),$$

where the second embedding was described in (2.3). This second embedding has the advantage that by restricting the representation first to the maximal semisimple Lie subalgebra $\mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2m}$, we get already a description of $\mathbb{V}(k\omega_{2m})$ as a sum of tensor products of $\mathfrak{sp}_{2m}$-modules, which will simplify the comparison with the decomposition of the Zhu algebra in theorem 1.1.

Step 2. It remains to determine the $Sp_{2m} \times Sp_{2m}$ structure of the representation $\mathrm{res}^{Sp_{4m}}_{Sp_{2m}\times Sp_{2m}} \mathbb{V}(k\omega_{2m})$. We use geometric methods. Let $P_{2m} \subset Sp_{4m}$ be the maximal parabolic subgroup associated with the weight ω_{2m} and set $Y = Sp_{4m}/P_{2m} \subset \mathbb{P}(\mathbb{V}(\omega_{2m}))$. We denote by $\widehat{Y} \subset \mathbb{V}(\omega_{2m})$ the affine cone over the projective variety.

The group Sp_{4m} acts on the affine variety $\widehat{Y}$ and hence on its coordinate ring $\mathbb{C}[\widehat{Y}]$. As Sp_{4m}-module, this ring is the direct sum:

$$\mathbb{C}[\widehat{Y}] = \bigoplus_{\ell \geqslant 0} \mathbb{V}(\ell\omega_{2m}).$$

The ring $\mathbb{C}[\widehat{Y}]$ is naturally graded with $\mathbb{V}(\ell\omega_{2m})$ as ℓth graded component.

Let $U \subset G = Sp_{2m} \times Sp_{2m}$ be the unipotent radical of a Borel subgroup. In a representation of G the U-fixed vectors are sums of highest weight vectors. The ring $\mathbb{C}[\widehat{Y}]^U$ of U-invariant vectors completely determines the structure of $\mathbb{C}[\widehat{Y}]$ as G-representation. □

Proposition 4.2. *The ring of U-invariant functions*

$$\mathbb{C}[\widehat{Y}]^U = \bigoplus_{k \geqslant 0} \mathbb{V}(k\omega_{2m})^U$$

is a polynomial ring generated by its degree 1 elements of weight $\omega_i \otimes \omega_i$ for $i = 0, \ldots, m$, where ω_0 denotes the trivial weight.

The proof will be given in step 4. As an immediate consequence, we get

Corollary 4.3.

$$\left(\mathrm{res}^{Sp_{4m}}_{Sp_{2m}\times Sp_{2m}} \mathbb{V}(k\omega_{2m})\right) = \bigoplus_{\substack{\lambda = \sum_{i=1}^m a_i\omega_i \\ \sum a_i \leqslant k}} V(\lambda) \otimes V(\lambda).$$

Step 3. **Proof of theorem 4.1.** The theorem follows now from corollary 4.3 together with lemma 2.4, theorem 1.1 and (2.1).

Step 4. It remains to prove proposition 4.2. The first step in this direction is as follows.

Proposition 4.4. *The action of $G = Sp_{2m} \times Sp_{2m}$ on $Y = Sp_{4m}/P_{2m}$ is spherical, i.e. a Borel subgroup of G has a dense orbit in Y.*

Proof. We use the local structure theorem [BLV], which states the following.

Let G be a connected complex reductive algebraic group. Suppose Y is a normal G-variety and $y \in Y$ is such that the stabilizer G_y of y is a parabolic subgroup of G, i.e. the orbit $G \cdot y$ is a projective variety. Let Q be a parabolic subgroup opposite to G_y. Denote by Q^u the unipotent radical of Q and set $L := G_y \cap Q$. □

Theorem 4.5 ([BLV]). *There exists a locally closed, affine subvariety Z of Y such that Z contains y and is stable under the action of L, $Q^u.Z$ is open in Y, and the canonical map $Q^u \times Z \to Q^u \cdot Z$ is an isomorphism of varieties.*

In our situation, we have $Y = Sp_{4m}/P_{2m}$, $y = \overline{1}$ is the class of the identity, G_y is $R = P_m \times P_m$, where $P_m \subset Sp_{2m}$ is the maximal parabolic associated with the fundamental weight ω_m. Let Q be the opposite parabolic subgroup, then $L = R \cap Q = GL_m \times GL_m$.

Denote by Q^u the unipotent radical of Q and let

$$O = G.\overline{1} \simeq (Sp_{2m} \times Sp_{2m})/(P_m \times P_m) = G/R$$

be the closed orbit in Y. If we apply the local structure theorem to this situation, then we may assume that Z is smooth since Y is smooth. Moreover, the action of G on Y is spherical if the action of L on Z is spherical. Consider the normal bundle $\mathcal{N}$ of O in X with fiber N at the coset $\overline{1}$ of the identity in G/R. Then N is isomorphic (as L-module) to the tangent space $T_y Z$ of the L-fixed point y. It follows now by Luna's slice theorem that the action of L on Z is spherical if and only if the action on N is spherical. Now as the representation of $L = GL_m \times GL_m$ we have that $N = \mathbb{C}^m \otimes \mathbb{C}^m$, which is a spherical action.

Let $\widehat{Y}$ be the affine cone over Y. Since the coordinate ring $\mathbb{C}[\widehat{Y}]$ of the affine cone over $Y = Sp_{4m}/P_{2m}$ is a unique factorization ring, it follows (see for example [L1], lemma 1)

Corollary 4.6. *The ring of U-invariant functions:*

$$\mathbb{C}[\widehat{Y}]^U = \bigoplus_{\ell \geqslant 0} \mathbb{V}(\ell\omega_{2m})^U$$

is a polynomial ring.

Step 5. To finish the proof of proposition 4.2, we have to show that the generators of $\mathbb{C}[\widehat{Y}]^U$ are of degree 1 and have the desired weights.

To calculate the $Sp_{2m} \times Sp_{2m}$ character, recall that (see for example [FH], theorem 17.5)

$$\mathrm{char}\mathbb{V}(\omega_{2m}) = \mathrm{char}\Lambda^{2m}(\mathbb{C}^{2m} \oplus \mathbb{C}^{2m}) - \mathrm{char}\Lambda^{2m-2}(\mathbb{C}^{2m} \oplus \mathbb{C}^{2m}). \tag{4.1}$$

More generally, for all $p = 1, \ldots, 2m$ one has

$$\mathrm{char}\mathbb{V}(\omega_p) = \mathrm{char}\Lambda^p(\mathbb{C}^{2m} \oplus \mathbb{C}^{2m}) - \mathrm{char}\Lambda^{p-2}(\mathbb{C}^{2m} \oplus \mathbb{C}^{2m}). \tag{4.2}$$

Using (4.1) and (4.2), one easily verifies that, as $Sp_{2m} \times Sp_{2m}$ module, we have

$$\mathbb{V}(\omega_{2m}) = \mathbb{C} \oplus V(\omega_1) \otimes V(\omega_1) \oplus V(\omega_2) \otimes V(\omega_2) \oplus \cdots \oplus V(\omega_m) \otimes V(\omega_m).$$

Let $f_0, \ldots, f_m \in \mathbb{V}(\omega_{2m})$ be the highest weight vectors for the $Sp_{2m} \times Sp_{2m}$-action, where f_0 is a $Sp_{2m} \times Sp_{2m}$ invariant function and f_i is of weight $\omega_i \otimes \omega_i$ for $i \geqslant 1$. Since $\mathbb{C}[\widehat{Y}]^U$ is a polynomial ring, the grading and weights imply that these elements are algebraically independent. In addition, if these elements do not generate the ring, then necessarily the Krull dimension $\dim \mathbb{C}[\widehat{Y}]^U > m + 1$. So proposition 4.2 is a consequence of the following lemma.

Lemma 4.7. *The Krull dimension* $\dim \mathbb{C}[\widehat{Y}]^U = m + 1$.

Proof. Since $\mathbb{C}[\widehat{Y}]$ is a UFD, U is connected and has no non-trivial characters, $\mathbb{C}(\widehat{Y})^U$ is the quotient field of $\mathbb{C}[\widehat{Y}]^U$. By Rosenlicht's theorem [R], generic orbits of an arbitrary action of a linear algebraic group on an irreducible algebraic variety are separated by rational invariants, which implies in our case that $\mathrm{trdeg}\mathbb{C}(\widehat{Y})^U = \dim \widehat{Y} - \dim(\text{generic orbit})$.

The maximal unipotent subgroup of $GL_m \times GL_m$ acts freely on an open subset of $\mathbb{C}^m \otimes \mathbb{C}^m$, so the local structure theorem (step 4) shows that U operates freely on an open subset of Y and hence does so on $\widehat{Y}$. Since the co-dimension of a generic U-orbit in $\widehat{Y}$ is $m+1$, this finishes the proof of the lemma and hence of proposition 4.2. □

5. The orthogonal case

In this section, we consider the case of orthogonal algebras $\mathfrak{so}_{2m}$ and $\mathfrak{so}_{2m+1}$. Our goal is to answer questions (i) and (ii) from section 1. Unfortunately, at the moment we are not able to answer these questions completely. The reason is that the C_2-algebra $A_{[2]}(\mathfrak{so}_n; k)$ is bigger than the representation $V(k\omega_n)$ of $\mathfrak{so}_{2n}$. Therefore, we can control only a certain quotient of the C_2-algebra. The details are given below.

5.1. Even orthogonal case: the setup

For the enumeration of fundamental weights we use the same notation as in [B]. Let $\omega_1, \dots, \omega_m$ be the set of fundamental weights of $\mathfrak{so}_{2m}$. The highest root is $\theta = \omega_2$, and for $\lambda = \sum_{i=1}^m a_i\omega_i$ the condition $(\lambda, \theta) \leqslant k$ can be reformulated as

$$a_1 + \sum_{i=2}^{m-2} 2a_i + a_{m-1} + a_m \leqslant k.$$

We also note that for even m all representations are self-dual and for odd m spin representations are dual to each other $V(\omega_{m-1})^* \simeq V(\omega_m)$. Thus, theorem 1.1 for $\mathfrak{g} = \mathfrak{so}_{2m}$ can be reformulated as

$$A(\mathfrak{so}_{2m}; k) \simeq \bigoplus_{\substack{\lambda=\sum_{i=1}^m a_i\omega_i \\ a_1+\sum_{i=2}^{m-2} 2a_i+a_{m-1}+a_m\leqslant k}} V_\lambda \otimes V_\lambda^*,$$

where $V(\lambda)^* = V(\lambda)$ for even m and

$$V\left(\sum_{i=1}^m a_i\omega_i\right)^* = V\left(\sum_{i=1}^{m-2} a_i\omega_i + a_{m-1}\omega_m + a_m\omega_{m-1}\right)$$

for odd m.

Let $\mathfrak{p}_{2m} \subset \mathfrak{so}_{4m}$ be the maximal parabolic Lie subalgebra associated with the fundamental weight ω_{2m}. We fix a Levi decomposition $\mathfrak{p}_{2m} = \mathfrak{l} \oplus \mathfrak{n}$ and $\mathfrak{g} = \mathfrak{p}_{2m} \oplus \mathfrak{n}^-$, where $\mathfrak{l} \simeq \mathfrak{gl}_{2m}$ as the Lie algebra and $\mathfrak{n} \simeq \Lambda^2(\mathbb{C}^{2m})$, respectively, $\mathfrak{n}^- \simeq \Lambda^2(\mathbb{C}^{2m})^*$ as $\mathfrak{l} = \mathfrak{gl}_{2m}$-module. The restriction of the $\mathfrak{gl}_{2m}$-representation on $\mathfrak{n}$ (respectively $\mathfrak{n}^-$) to the subalgebra $\mathfrak{so}_{2m} \subset \mathfrak{gl}_{2m}$ remains irreducible; it is the adjoint representation of the orthogonal Lie algebra. Summarizing we have

Lemma 5.1. *As $\mathfrak{gl}_{2m}$-module we have isomorphisms $\mathfrak{n} \simeq \Lambda^2(\mathbb{C}^{2m})$, $\mathfrak{n}^- \simeq \Lambda^2(\mathbb{C}^{2m})^*$, and as $\mathfrak{so}_{2m}$-module we have isomorphisms $\mathfrak{n} \simeq \mathfrak{n}^- \simeq \mathfrak{so}_{2m}$.*

In the following, we always assume that for $\ell \in \mathbb{N}$ the orthogonal group $SO_{2\ell}$ is defined to be the group leaving invariant the symmetric form on $\mathbb{C}^{2\ell}$ defined by the $2\ell \times 2\ell$ matrix:

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdot^{\cdot^{\cdot}} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For an $m \times m$ matrix A let A^{nt} be the transpose as in section 2. The Lie algebra of the orthogonal group SO_{2m} can then be described as the following set of matrices:

$$\mathfrak{so}_{2m} = \left\{ \begin{pmatrix} A & B \\ C & -A^{nt} \end{pmatrix} \middle| A, B, C \in M_m, B = -B^{nt}, C = -C^{nt} \right\},$$

with the maximal torus $\mathfrak{t} = \mathrm{diag}(t_1, \ldots, t_m, -t_m \ldots, -t_1)$ and the Borel subalgebra being the upper triangular matrices of the form above.

The Lie algebra of the orthogonal group $SO_{2m} \subset GL_{2m}$ embedded in the Levi subgroup $GL_{2m} \subset SO_{4m}$ can be seen as the set of matrices of the following form:

$$\mathfrak{so}_{2m} = \left\{ \begin{pmatrix} A & B & 0 & 0 \\ C & -A^{nt} & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & -A^{nt} \end{pmatrix} \middle| \begin{matrix} A, B, C \in M_m, \\ B = -B^{nt} \\ C = -C^{nt} \end{matrix} \right\} \subset \mathfrak{so}_{4m}. \tag{5.1}$$

There is also a maximal reductive sub-Lie algebra of type $\mathtt{D}_m + \mathtt{D}_m$ sitting inside $\mathfrak{so}_{4m}$ in the following way:

$$\mathfrak{so}_{2m} \oplus \mathfrak{so}_{2m} = \left\{ \begin{pmatrix} K & 0 & 0 & L \\ 0 & X & Y & 0 \\ 0 & Z & -X^{nt} & 0 \\ M & 0 & 0 & -K^{nt} \end{pmatrix} \middle| \begin{matrix} K, L, M, X, Y, Z \in M_m \\ L = -L^{nt}, M = -M^{nt} \\ Y = -Y^{nt}, Z = -Z^{nt} \end{matrix} \right\}.$$

Let $\mathbb{I}_m$ denote the $m \times m$ identity matrix and let J' be the $2m \times 2m$ matrix of the form

$$J' = \begin{pmatrix} 0 & \mathbb{I}_m \\ \mathbb{I}_m & 0 \end{pmatrix}.$$

Inside the Lie subalgebra $\mathfrak{so}_{2m} \oplus \mathfrak{so}_{2m}$ we have a diagonally embedded orthogonal Lie algebra $\Delta(\mathfrak{so}_{2m})$, where

$$\Delta : \mathfrak{so}_{2m} \hookrightarrow \mathfrak{so}_{2m} \oplus \mathfrak{so}_{2m}, \qquad Q \mapsto (Q, J'QJ'^{-1}), \tag{5.2}$$

which has the same maximal torus as the embedded orthogonal Lie subalgebra $\mathfrak{so}_{2m}$ described in (5.1). As a consequence, we see

Lemma 5.2. *For a $\mathfrak{so}_{4m}$-representation $\mathbb{V}(\lambda)$ let $\mathrm{res}^{\mathfrak{so}_{4m}}_{\mathfrak{so}_{2m}} \mathbb{V}(\lambda)$ be the $\mathfrak{so}_{2m}$-representation obtained via the embedding in (5.1) and denote by*

$$\mathrm{res}^{\mathfrak{so}_{4m}}_{\Delta(\mathfrak{so}_{2m})} \mathbb{V}(\lambda)$$

the $\mathfrak{so}_{2m}$-representation obtained via the embedding in (5.2).

Then $\mathrm{res}^{\mathfrak{so}_{4m}}_{\mathfrak{so}_{2m}} \mathbb{V}(\lambda) \simeq \mathrm{res}^{\mathfrak{so}_{4m}}_{\Delta(\mathfrak{so}_{2m})} \mathbb{V}(\lambda)$.

5.2. The Lie algebra $\mathfrak{so}_{4m}$ and a quotient of $A_{[2]}(\mathfrak{so}_{2m}; k)$

We fix the standard maximal torus (diagonal matrices) and Borel subalgebra (upper triangular matrices) for $\mathfrak{gl}_{2m}$; then the maximal torus and the Borel subalgebra of $\mathfrak{so}_{2m}$ and $\mathfrak{gl}_{2m}$ are contained in each other. Let θ be the maximal root in the root system of $\mathfrak{so}_{2m}$, we identify the module $\mathfrak{so}_{2m}$ again with $\mathfrak{n}^-$ (lemma 5.1). Fix a highest root vector x_θ, then x_θ is a highest weight vector for the adjoint action of $\mathfrak{so}_{2m}$ as well as for the irreducible action of $\mathfrak{gl}_{2m}$ on the same representation space. It follows that

$$x_\theta^{k+1} \in S^\bullet(\mathfrak{so}_{2m}) = S^\bullet(\Lambda^2(\mathbb{C}^{2m}))^* = S^\bullet(\mathfrak{n}) \tag{5.3}$$

is a highest weight vector of weight $(k+1)\omega_2$ for the action of the orthogonal group SO_{2m} and of weight $-(k+1)(\epsilon_{2m-1} + \epsilon_{2m})$ for the general linear group GL_{2m}. One can check easily the following connection between x_θ and the root vectors for the Lie algebra $\mathfrak{so}_{4m}$.

Lemma 5.3. *Let $X_{-\alpha_{2m}} \in \mathfrak{n} \subset \mathfrak{so}_{4m}$ be a root vector for the negative of the simple root α_{2m} of the root system of $\mathfrak{so}_{4m}$. With respect to the embedding in (5.1), $X_{-\alpha_{2m}}$ is a weight vector for the Lie algebra $\mathfrak{so}_{2m}$ of weight θ.*

To distinguish between the highest weight representations of the different groups, we write $V(\lambda)$ for the $\mathfrak{so}_{2m}$-representations, $\mathbb{V}(\lambda)$ for the $\mathfrak{gl}_{2m}$-representations and $\mathbb{W}(\lambda)$ for the $\mathfrak{so}_{4m}$-representations of highest weight λ (whenever this makes sense).

The irreducible $\mathfrak{gl}_{2m}$-module $U(\mathfrak{gl}_{2m})\circ x_\theta^{k+1}$ generated by x_θ^{k+1} is of weight $-(k+1)(\epsilon_{2m-1}+\epsilon_{2m})$, and hence does not remain irreducible when restricted to $\mathfrak{so}_{2m}$, i.e. we have the following sequence of inclusions of vector spaces:

$$U(\mathfrak{gl}_{2m}) \circ X_{-\alpha_{2m}}^{k+1} = U(\mathfrak{gl}_{2m}) \circ x_\theta^{k+1} \supset U(\mathfrak{so}_{2m}) \circ x_\theta^{k+1} = U(\mathfrak{so}_{2m}) \circ X_{-\alpha_{2m}}^{k+1}. \tag{5.4}$$

Let $\mathbb{W}(k\omega_{2m})$ be the irreducible Spin_{4m}-module of highest weight $k\omega_{2m}$. The nilpotent radical $\mathfrak{n}$ of $\mathfrak{p}_{2m}$ is Abelian (since ω_{2m} is a cominuscule weight, see for example [FFL]). Recall the following isomorphism of $\mathfrak{l}$-modules (see [FFL], lemma 3.1):

$$\mathbb{W}(k\omega_{2m}) \otimes \mathbb{C}_{-k\omega_{2m}} \simeq S^\bullet(\mathfrak{n}^-)\big/\big\langle U(\mathfrak{l}) \circ x_\theta^{k+1}\big\rangle,$$

where $\langle\cdots\rangle$ denotes the ideal generated by the corresponding subspace. Combining this isomorphism with (5.4), we get as a consequence the following morphisms of $\mathfrak{gl}_{2m}$-modules respectively $\mathfrak{so}_{2m}$-modules. In particular, we obtain a quotient of the C_2-algebra $A_{[2]}(\mathfrak{so}_{2m}; k)$ as a $\mathfrak{so}_{4m}$-representation:

Lemma 5.4.

$$\begin{aligned} A_{[2]}(\mathfrak{so}_{2m}; k) &= S^\bullet(\mathfrak{so}_{2m})\big/\big\langle U(\mathfrak{so}_{2m}) \circ x_\theta^{k+1}\big\rangle \\ &\to S^\bullet(\Lambda^2\mathbb{C}^{2m})^*\big/\big\langle U(\mathfrak{gl}_{2m}) \circ x_\theta^{k+1}\big\rangle \\ &= S^\bullet(\mathfrak{n}^-)\big/\big\langle U(\mathfrak{gl}_{2m}) \circ X_{\alpha_{2m}}^{k+1}\big\rangle \\ &\simeq \mathbb{W}(k\omega_{2m}) \otimes \mathbb{C}_{-k\omega_{2m}}. \end{aligned}$$

Remark 5.5. Of course, one can use the representation $\mathbb{W}(k\omega_{2m-1})$ instead of $\mathbb{W}(k\omega_{2m})$. It turns out that in order to compare Zhu's algebra and the C_2-algebra of type D_m one has to use the first representation for odd m and the second one for even m. We work out in details the even m case. The odd case is considered in subsection 5.4.

5.3. The $\mathfrak{so}_{2m} \times \mathfrak{so}_{2m}$-decomposition of $\mathbb{W}(k\omega_{2m})$

In the following, we investigate the decomposition of $\mathbb{W}(k\omega_{2m})$ as $\mathfrak{so}_{2m}$-module. By lemma 5.2 it suffices to describe the $\mathfrak{so}_{2m} \times \mathfrak{so}_{2m}$-module structure.

We proceed as in the symplectic case and use geometric methods: let $P_{2m} \subset \mathrm{Spin}_{4m}$ be the maximal parabolic subgroup associated with the weight ω_{2m} and set $Y = \mathrm{Spin}_{4m}/P_{2m} \subset \mathbb{P}(\mathbb{W}(k\omega_{2m}))$. We denote by $\widehat{Y} \subset \mathbb{W}(k\omega_{2m})$ the affine cone over the projective variety.

The group Spin_{4m} acts on the affine variety $\widehat{Y}$ and hence on its coordinate ring $\mathbb{C}[\widehat{Y}]$. As Spin_{4m}-module, this ring is the direct sum:

$$\mathbb{C}[\widehat{Y}] = \bigoplus_{\ell \geqslant 0} \mathbb{W}(\ell\omega_{2m}).$$

The ring $\mathbb{C}[\widehat{Y}]$ is naturally graded with $\mathbb{W}(\ell\omega_{2m})$ as ℓth graded component. Let $U \subset G = \mathrm{Spin}_{2m} \times \mathrm{Spin}_{2m}$ be the unipotent radical of a Borel subgroup.

Proposition 5.6. *The ring of U-invariant functions*

$$\mathbb{C}[\widehat{Y}]^U = \bigoplus_{k\geqslant 0} \mathbb{W}(k\omega_{2m})^U$$

is a polynomial ring generated by its degree 1 elements of weight $\omega_m \otimes \omega_m$ and $\omega_{m-1} \otimes \omega_{m-1}$ and the degree 2 elements $\omega_i \otimes \omega_i$ for $i = 0, \ldots, m-2$, where ω_0 denotes the trivial weight.

As an immediate consequence, we get

Corollary 5.7.

$$\mathrm{res}^{\mathrm{Spin}_{4m}}_{\mathrm{Spin}_{2m}\times\mathrm{Spin}_{2m}} \mathbb{W}(k\omega_{2m}) = \bigoplus_{\substack{\lambda=\sum_{i=1}^m a_i\omega_i \\ (\sum_{i=1}^{m-2} 2a_i)+a_{m-1}+a_m\leqslant k \\ k-a_{m-1}-a_m\equiv 0 \pmod 2}} V(\lambda)\otimes V(\lambda).$$

It remains to prove proposition 5.6. A first step in this direction is the following.

Proposition 5.8. *The action of $G = \mathrm{Spin}_{2m} \times \mathrm{Spin}_{2m}$ on $Y = \mathrm{Spin}_{4m}/P_{2m}$ is spherical, i.e. a Borel subgroup of G has a dense orbit in Y.*

Proof. As before we use the local structure theorem [BLV]. In our situation, we have $Y = \mathrm{Spin}_{4m}/P_m$, $y = \overline{1}$ is the class of the identity, G_y is $R = P_m \times P_m$, where $P_m \subset \mathrm{Spin}_{2m}$ is the maximal parabolic associated with the fundamental weight ω_m. Let Q be the opposite parabolic subgroup, then $\mathfrak{l} = \mathfrak{gl}_m \times \mathfrak{gl}_m$.

Denote by Q^u the unipotent radical of Q and let

$$O = G.\overline{1} \simeq (\mathrm{Spin}_{2m} \times \mathrm{Spin}_{2m})/(P_m \times P_m) = G/R$$

be the closed orbit in Y. If we apply the local structure theorem to this situation, then we may assume that Z is smooth since Y is smooth. Moreover, the action of G on Y is spherical if the action of L on Z is spherical. Consider the normal bundle $\mathcal{N}$ of O in X with fiber N at the coset $\overline{1}$ of the identity in G/R. Then N is isomorphic (as L-module) to the tangent space T_yZ of the L-fixed point y. It follows now by Luna's slice theorem that the action of L on Z is spherical if and only if the action on N is spherical. Now as representation for the Lie algebra $\mathfrak{l} = \mathfrak{gl}_m \oplus \mathfrak{gl}_m$, we get the action on $N = \mathbb{C}^m \otimes \mathbb{C}^m$, which is a spherical action. □

Let $\widehat{Y}$ be the affine cone over Y. Since the coordinate ring $\mathbb{C}[\widehat{Y}]$ of the affine cone over $Y = \mathrm{Spin}_{4m}/P_{2m}$ is a unique factorization ring, it follows

Corollary 5.9. *The ring of U-invariant functions*

$$\mathbb{C}[\widehat{Y}]^U = \bigoplus_{\ell\geqslant 0} \mathbb{W}(\ell\omega_{2m})^U$$

is a polynomial ring.

Proof of proposition 5.6. To finish the proof of proposition 5.6 we have to show that the generators of $\mathbb{C}[\widehat{Y}]^U$ are of the desired degrees and weights.

To calculate the $\mathfrak{so}_{2m} \oplus \mathfrak{so}_{2m}$ character, recall that (see for example [FH], theorem 19.2) for the Spin_{4m}-modules, we have

$$\Lambda^{2m}\mathbb{C}^{4m} = \Lambda^{2m}(\mathbb{C}^{2m} \oplus \mathbb{C}^{2m}) = \mathbb{W}(2\omega_{2m}) \oplus \mathbb{W}(2\omega_{2m-1}).$$

Using the decomposition $\mathbb{C}^{4m} = (\mathbb{C}^{2m} \oplus \mathbb{C}^{2m})$, with this description one verifies easily that, as $\mathfrak{so}_{2m} \oplus \mathfrak{so}_{2m}$-module, we have

$$\begin{aligned}\mathrm{res}^{\mathfrak{so}_{4m}}_{\mathfrak{so}_{2m}\oplus\mathfrak{so}_{2m}} \mathbb{W}(2\omega_{2m}) = \mathbb{C} \oplus V(\omega_1) \otimes V(\omega_1) \oplus \cdots \oplus V(\omega_{m-2}) \otimes V(\omega_{m-2})\\ \oplus V(\omega_{m-1} + \omega_m) \otimes V(\omega_{m-1} + \omega_m)\\ \oplus V(2\omega_{m-1}) \otimes V(2\omega_{m-1}) \oplus V(2\omega_m) \otimes V(2\omega_m).\end{aligned}$$

For the fundamental representation, one computes

$$\mathrm{res}^{\mathfrak{so}_{4m}}_{\mathfrak{so}_{2m}\oplus\mathfrak{so}_{2m}} \mathbb{W}(\omega_{2m}) = V(\omega_{m-1}) \otimes V(\omega_{m-1}) \oplus V(\omega_m) \otimes V(\omega_m).$$

Let $f_0, \ldots, f_m \in \mathbb{W}(\omega_{2m}) \oplus \mathbb{W}(2\omega_{2m})$ be the highest weight vectors for the $\mathfrak{so}_{2m} \oplus \mathfrak{so}_{2m}$-action, where f_0 is a $\mathfrak{so}_{2m} \oplus \mathfrak{so}_{2m}$-invariant function of degree 2, $f_1, \ldots, f_{m-2}$ are of weight $\omega_i \otimes \omega_i$ for $i = 1, \ldots, m-2$ and of degree 2, and f_{m-1}, f_m are of degree 1 and of weight $\omega_{m-1} \otimes \omega_{m-1}$ and $\omega_m \otimes \omega_m$, respectively.

The collection of functions $f_0, \ldots, f_m, f_{m-1}^2, f_{m-1} f_m, f_m^2$ is a basis for the subspace of the highest weight vectors in $\mathbb{W}(\omega_{2m}) \oplus \mathbb{W}(2\omega_{2m})$.

Since $\mathbb{C}[\widehat{Y}]^U$ is a polynomial ring, the grading and the weights imply that the elements $f_0, \ldots, f_m$ are algebraically independent. In addition, if these elements do not generate the ring, then necessarily $\dim \mathbb{C}[\widehat{Y}]^U > m+1$. So proposition 5.6 is a consequence of the following lemma, which is proved along the same lines as lemma 4.7. □

Lemma 5.10. $\dim \mathbb{C}[\widehat{Y}]^U = m + 1$.

5.4. The dual realization

Recall that for odd m not all representations of $\mathfrak{so}_{2m}$ are self-dual. Therefore, corollary 5.7 does not allow us to compare Zhu's algebras and the C_2-algebras. In fact, to handle this problem, we consider the weight ω_{2m-1} of $\mathfrak{so}_{4m}$ (instead of ω_{2m}). This weight is also cominuscule and thus everything works for ω_{2m-1} as well. Below we formulate the analogs of proposition 5.6 and corollary 5.7.

Fix an odd m. Let $P_{2m-1} \subset \mathrm{Spin}_{4m}$ be the maximal parabolic subgroup associated with the weight ω_{2m-1} and set $Y' = \mathrm{Spin}_{4m}/P_{2m-1} \subset \mathbb{P}(\mathbb{W}(k\omega_{2m-1}))$. We denote by $\widehat{Y}' \subset \mathbb{W}(k\omega_{2m-1})$ the affine cone over the projective variety.

Proposition 5.11. *The ring of U-invariant functions*

$$\mathbb{C}[\widehat{Y}']^U = \bigoplus_{k\geqslant 0} \mathbb{W}(k\omega_{2m-1})^U$$

is a polynomial ring generated by its degree 1 elements of weight $\omega_m \otimes \omega_{m-1}$ and $\omega_{m-1} \otimes \omega_m$ and the degree 2 elements $\omega_i \otimes \omega_i$ for $i = 0, \ldots, m-2$, where ω_0 denotes the trivial weight.

As an immediate consequence, we get

Corollary 5.12.

$$\mathrm{res}^{\mathrm{Spin}_{4m}}_{\mathrm{Spin}_{2m}\times\mathrm{Spin}_{2m}} \mathbb{W}(k\omega_{2m-1}) = \bigoplus_{\substack{\lambda=\sum_{i=1}^m a_i\omega_i\\ (\sum_{i=1}^{m-2} 2a_i)+a_{m-1}+a_m\leqslant k\\ k-a_{m-1}-a_m\equiv 0 \pmod 2}} V(\lambda) \otimes V(\lambda)^*.$$

5.5. Comparison

Summarizing, we obtain the following.

Proposition 5.13. *We have an isomorphism of $\mathfrak{so}_{2m}$ modules*

$$S^\bullet(\mathfrak{so}_{2m})\big/\langle U(\mathfrak{gl}_{2m})\circ e_\theta^{k+1}\rangle \simeq \bigoplus_{\substack{\lambda=\sum_{i=1}^m a_i\omega_i\\ (\sum_{i=1}^{m-2}2a_i)+a_{m-1}+a_m\leqslant k\\ k-a_{m-1}-a_m\equiv 0 \pmod 2}} V(\lambda)\otimes V(\lambda)^*, \tag{5.5}$$

providing a surjection of $A_{[2]}(\mathfrak{so}_{2m};k)$ to the righthand side of (5.5).

5.6. The odd orthogonal case

In this subsection, we consider the case $\mathfrak{g}=\mathfrak{so}_{2m+1}$. All the constructions as above are valid in this case as well, so we only formulate the final result in the following proposition.

Proposition 5.14. *There exists a surjection of $\mathfrak{so}_{2m+1}$ modules:*

$$A_{[2]}(\mathfrak{so}_{2m+1};k)\to \bigoplus_{\substack{\lambda=\sum_{i=1}^m a_i\omega_i\\ (\sum_{i=1}^{m-1}2a_i)+a_m\leqslant k\\ k-a_m\equiv 0 \pmod 2}} V(\lambda)\otimes V(\lambda)^*.$$

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